Week 14 disc 104

Wednesday, April 28, 2021 11:04 AM

Problem 1. *Find all values of α that make the following matrix singular*

$$A = \begin{pmatrix} 1 & -1 & \alpha \\ 2 & 2 & 1 \\ 0 & \alpha & -\frac{3}{2} \end{pmatrix}$$

Find α s.t. $\det(A) = 0$

$$\det(A) = (1)(-3 - \alpha) - (2)\left(\frac{3}{2} - \alpha^2\right)$$

$$= -3 \cancel{+}\alpha - 3 + 2\alpha^2$$

$$= 2\alpha^2 \cancel{+}\alpha - 6$$

$$\alpha = \frac{+1 \pm \sqrt{1+48}}{4} = \frac{+1 \pm 7}{4} = \left\{ \cancel{-\tfrac{3}{2}}, \cancel{+}2 \right\}$$

$3/2$

$$-\alpha(1 - 2\alpha) - \frac{3}{2}(2 + 2)$$

Problem 2. *Prove that AB is nonsingular if and only if both A and B are nonsingular*

Problem 3. *It turns out that if A depends on a parameter t in a C^1 way (the components of A are C^1 functions of t), then $\frac{d}{dt}\log|\det A(t)| = Tr(A^{-1}\frac{d}{dt}A)$. Verify this for the matrix:*

$$A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$$

2) AB nonsingular $\iff \det(AB) \neq 0$

$\iff \det(A)\det(B) \neq 0$

$\iff \det(A) \neq 0$ & $\det(B) \neq 0$

$\iff A^{-1}$ & B^{-1} exist.

3) $LHS = \frac{d}{dt}\log|ad - bc| = \frac{1}{ad-bc}(a'd + ad' - b'c - bc')$

$$RHS = Tr\left(\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right)$$

$$= \frac{1}{ad-bc} Tr\begin{pmatrix} da' - bc' & * \\ * & -b'c + ad' \end{pmatrix} = \frac{1}{ad-bc}\left[da' - bc' - b'c + ad'\right]$$

Problem 4. *Solve the linear system of equations:*

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 2 & 1 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}} = \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}$$

Let $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -1 \\ & 4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\quad \begin{pmatrix} 2 & 1 & -1 \\ 0 & 4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Let $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}$

$y_1 = 1$

$y_2 = 2y_1 = 2$

$y_3 = -5 - 3 = -8$

$\begin{pmatrix} 2 & 1 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -8 \end{pmatrix}$

$x_3 = -8/5$

$x_2 = \dfrac{(2 - 2x_3)}{4}$

$x_1 = \dfrac{(1 - x_2 + x_3)}{2}$

Problem 5. *Consider the matrix:*

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

Find the permutation matrix P so that PA can be factored into the product LU, where L is lower triangular with 1s on its diagonal and U is upper triangular.

$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 0 \\ 0 & 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}$ ← we need to permute row 2 & 3

$\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$

$\boxed{P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$

Problem 6. *Determine which of the following matrices are symmetric, singular, strictyl diagonally dominant, positive definite:*

1. $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$
2. $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 4 \end{pmatrix}$
3. $\begin{pmatrix} 4 & 2 & 6 \\ 3 & 0 & 7 \\ -2 & -1 & -3 \end{pmatrix}$
4. $\begin{pmatrix} 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 \\ 9 & 11 & 1 & 0 \\ 5 & 4 & 1 & 1 \end{pmatrix}$

Symmetric: $\begin{pmatrix} a & c \\ c & b \end{pmatrix}$

Singular: $\det(A) = 0$

Strictly diagonally dominant: $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$

Positive definite: $x^t A x > 0 \quad \forall \; x \neq 0$

equivalently: all eigenvalues are positive.

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(1) $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$

Sym ✓

$\det(A) = 6 - 1 = 5 \neq 0$

SDD ✓

Positive definite. ✓ $0 = \det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} \Rightarrow$ get $\lambda < 0$

$(x \;\; y) \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (x \;\; y) \begin{pmatrix} 2x + y \\ x + 3y \end{pmatrix} = \boxed{2x^2 + xy + xy + 3y^2} \overset{?}{>} 0$

$$(x\ \ y)\begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = (x\ \ y)\begin{pmatrix}2x + y\\ x + 3y\end{pmatrix} = \boxed{2x^2 + xy + xy + 3y^2} \overset{?}{>} 0$$

if $\lambda < 0 \implies \exists\, v$ s.t. $Av = \lambda v$

$$v^T A v = v^T(\lambda v) = \underbrace{\lambda}_{<0}\,\underbrace{\|v\|^2}_{>0} < 0$$

2) $\begin{pmatrix}2 & 1 & 0\\ 0 & 3 & 0\\ 1 & 0 & 4\end{pmatrix} = A$

- not sym.
- $\det(A) = 2 \cdot 12 + 0 = 24 \neq 0$
- SDD ✓
- Positive definite ✓ by equivale.

3) $\begin{pmatrix}4 & 2 & 6\\ 3 & 0 & 7\\ -2 & -1 & -3\end{pmatrix} = A$

- not sym.
- $\det(A) = 0$ singular
- Not SDD
- Not positive def.

4) $\begin{pmatrix}4 & 0 & 0 & 0\\ 6 & 7 & 0 & 0\\ 9 & 1 & 0 & 0\\ 5 & 4 & 1 & 1\end{pmatrix}$ Positive def ✓

$$\begin{bmatrix} 0 \cdots\cdots\cdots 0 & a_{nn}^{(n)} \end{bmatrix}$$

Example 2 **(a)** Determine the LU factorization for the matrix A in the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ -3 \\ 4 \end{bmatrix}.$$

(b) Then use the factorization to solve the system

$$\begin{aligned} x_1 + x_2 \quad\quad + 3x_4 &= 8, \\ 2x_1 + x_2 - x_3 + x_4 &= 7, \\ 3x_1 - x_2 - x_3 + 2x_4 &= 14, \\ -x_1 + 2x_2 + 3x_3 - x_4 &= -7. \end{aligned}$$

Solution **(a)** The original system was considered in Section 6.1, where we saw that the sequence of operations $(E_2 - 2E_1) \to (E_2)$, $(E_3 - 3E_1) \to (E_3)$, $(E_4 - (-1)E_1) \to (E_4)$, $(E_3 - 4E_2) \to (E_3)$, $(E_4 - (-3)E_2) \to (E_4)$ converts the system to the triangular system

$$\begin{aligned} x_1 + x_2 \quad\quad + 3x_4 &= 4, \\ -x_2 - x_3 - 5x_4 &= -7, \\ 3x_3 + 13x_4 &= 13, \\ -13x_4 &= -13. \end{aligned}$$

The multipliers m_{ij} and the upper triangular matrix produce the factorization

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} = LU.$$

(b) To solve

$$A\mathbf{x} = LU\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 14 \\ -7 \end{bmatrix},$$

we first introduce the substitution $\mathbf{y} = U\mathbf{x}$. Then $\mathbf{b} = L(U\mathbf{x}) = L\mathbf{y}$. That is,

$$L\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 14 \\ -7 \end{bmatrix}.$$

$$\begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ & & & \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & & & 0 \\ m_{11} & \ddots & & \\ m_{21} & & \ddots & \\ m_{n1} & \cdots & m_{n,n-1} & 1 \end{pmatrix}$$

$$m_{ji} = \frac{a_{ji}^{(i)}}{a_{ii}^{(i)}}$$

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}}_{A} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 3 \\ 0 & -4 \end{bmatrix}$$

$$\underline{R_2'} = R_2 - 2R_1$$

$$R_2 = R_2' + 2R_1$$

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = M \qquad M^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$MA = U$$

$$A = M^{-1}U$$

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

$$R_2 \to R_2 - 2R_1$$

$$R_3 \to R_3 - 3R_1$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

$$M_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$$M_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

$$M_2 M_1 A = U$$

$$M_1 A = M_2^{-1} U$$

$$A = M_1^{-1} M_2^{-1} U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix}$$

$$\uparrow \quad \uparrow$$